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Slow motion of a bubble in a viscoelastic fluid

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SUMMARY

The motion of a deformed spherical body in a fluid medium is significantly different from the motion of an undeformed spherical body in the same medium. It is shown in this work that a bubble moving in a viscoelastic fluid takes the shape

$$r = a + \frac{U_0 \eta_0}{\sigma a}(\lambda_1 - \lambda_2)\frac{(180 R^3 + 240 R^2 + 816 R + 672) P_2(\cos\theta)}{960(1+R)^3}$$

and so one must expect the dynamics of a bubble moving in a non-Newtonian fluid to be significantly different from that of a bubble moving in a purely viscous fluid.

1. Introduction

Between the works of Saito [10] and Taylor & Acrivos [11] there was little work done on the slow motion of drops and bubbles despite the fact that a clear understanding of their shape and motion is essential in solving various experimental and practical engineering problems such as extraction from liquid drops, atomisation of drops in fuel injection into internal combustion engines, motion of raindrops and so on. The scarcity of theoretical work in this field is not surprising considering the complexity of the problem. At the same time as one is solving for the motion inside and outside of the bubble one has to determine the shape of the bubble. Another contributory factor is the fact that there are various phenomena which cannot be explained on the basis of Stokes' equation. These phenomena are suppressed by the neglect of inertia forces in Stokes' equations. These inertia forces being non-linear cause great mathematical difficulties and in fact it wasn't until the work of Kaplun [7] that much progress has been made in studying the effect of these forces on the motion of particles. It wasn't unusual, on account of the difficulties involved, to approximate drops and bubbles to rigid spheres. When this is done the justification is usually based on the fact that when the inertia terms of the equations of motion are negligible the bubble will behave exactly as a perfectly spherical body, Saito [10].

In particular, Hocking [6] when calculating the collision efficiency of randroplets in non-freezing clouds neglected the internal circulation of the drops and approximated the drops to rigid spheres. While this approximation is justifiable in this case it will not hold always true as has been clearly demonstrated by Ajayi [1] who showed that if the ratio of the viscosities of the fluid external to the drop and the fluid internal to the drop is not very much less than unity one cannot neglect the fluidity of the drop.

It is of fundamental importance to know what factors affect the shape and thus the motion of bubbles. While it is true that a single bubble moving in a purely viscous fluid remains perfectly spherical one must ask whether a bubble moving in a fluid which is not purely viscous will also remain spherical. Similarly one would also like to know whether a bubble moving in the neighbourhood of another bubble in a purely viscous fluid also remains spherical.

Following the introduction of the method of matched asymptotic expansion, Taylor & Acrivos employed this method and demonstrated that while a single bubble remains perfectly spherical when Re, Reynold's number, is zero, the same bubble would take a shape approaching a spherical cap when Re is significant. In other words, an effect of inertia forces is to cause the deformation of a bubble. Chaffey, Brenner & Mason [3, 4] have shown that a deformable liquid sphere moving close to a plane wall is deformed and that there exists a force tending to push the bubble away from the wall.

Hestroni, Haber & Wacholder [5] using the method of reflection showed that a cylindrical wall causes a bubble moving in a purely viscous fluid contained in the cylinder to be deformed. While more recently Ajayi [1] who considered the hydrodynamic interaction between two bubbles moving slowly in a purely viscous fluid, found that whether the bubbles are moving side by side or following each other, the interaction between the bubbles is also a causation of the deformation of the bubbles. And lastly Ajayi [2] has demonstrated that the shape of a spherical bubble which rotates in a viscous fluid will depart from spherical. In each work cited above, apart from the one of Taylor & Acrivos, the analysis was based on Stokes' slow flow equation. But as pointed out earlier, the suppression of non-linear terms in Stokes' equation necessarily obscures some physical phenomena of interest. In fact it is this suppression of such terms that was probably responsible for the little progress made in the theoretical study of slow motion of bubbles since the work of Saito. Wohl & Rubinow [12] motivated by a desire to explain the observed axial accumulation of red blood cells in blood flowing through small arteries in the circulatory system found that a deformable liquid sphere moving in an unbounded steady parabolic flow experiences a force which is perpendicular to its direction of motion arising out of the interaction between the flow and the sphere deformation. In other words the effect of the deformation of the drop is to produce a lift force. It is pertinent to point out that this lift force is absent if the body is purely spherical. It is clear therefore that the dynamics of a deformed bubble is very different from that of an undeformed bubble. We recall the fact that a bubble moving in a purely viscous fluid remains perfectly spherical but we shall show here that when the bubble moves in a viscoelastic liquid it does get deformed and one must therefore expect bubbles to behave differently in a viscoelastic fluid as compared with a purely viscous liquid.

The basis of this work is the characteristic equation of a non-Newtonian fluid put forward by Oldroyd. The term bubble is used throughout the work but the results are equally valid for a drop.

2. Statement of the problem

We consider here the steady flow of a non-Newtonian fluid of infinite extent past a drop of purely viscous and incompressible fluid which is suspended in the non-Newtonian fluid. The viscosity of the bubble fluid is denoted by $\hat{\eta}_0$. In the quiescent state the bubble is supposed to be perfectly spherical and of radius a. We choose spherical coordinates (r, θ, ϕ) whose centre coincides with the centre of the spherical bubble. The fluid at infinity is supposed to have a velocity U_0 in the $\theta=0$ direction. The ambient fluid is characterised by the rheological equations of state, relating the stress tensor S_{ik} and the rate-of-strain tensor

$$E_{ik} \equiv \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) \tag{1}$$

of the form

$$S_{ik} = P_{ik} - P g_{ik}\,, \tag{2}$$

thus

$$P_{ik} + \lambda_1 \frac{\mathbb{D}}{\mathbb{D}T} P_{ik} + \mu_0 P_{jj} E_{ik} + \nu_1 P_{jl} E_{jl} g_{ik} = 2\eta_0 \left[E_{ik} + \lambda_2 \frac{\mathbb{D}}{\mathbb{D}T} E_{ik} + \nu_2 E_{jl} E_{jl} g_{ik} \right].$$

The derivative $\mathbb{D}/\mathbb{D}T$ is defined for any tensor A_{ik} by

$$\frac{\mathbb{D}}{\mathbb{D}T} A_{ik} = \frac{\partial A_{ik}}{\partial T} + u_j \frac{\partial A_{ij}}{\partial x_j} + w_{im} A_{mk} + w_{km} A_{im} - E_{im} A_{mk} - E_{km} A_{im}\,, \tag{4}$$

with

$$w_{ik} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_i} u_k - \frac{\partial}{\partial x_k} u_i\right) \tag{5}$$

where u_i denotes the velocity vector, g_{ik} the metric tensor and P an arbitrary isotropic pressure which can be superposed on an incompressible fluid element without affecting the flow. λ_1, λ_2, μ_0, ν_1, ν_2 are constants having the dimension of time and η_0 is a constant having the dimension of viscosity.

Viscoelastic liquids are a class of liquids which exhibit certain non-Newtonian properties such as the Weissenberg climbing effect. They also possess a variable apparent viscosity in simple shearing, decreasing with increasing rate of shear from a limiting value η_0 at low rates to a lower limiting value η_1 at high rates, and they have a distribution of normal stresses corresponding to an extra simple tension along the streamlines in many types of steady shearing flow. Such fluids are characterised by only three constants, a coefficient of viscosity, a relaxation time and a retardation time, for flows at small shear stresses. Equations (2) and (3) were put forward by Oldroyd [9] to simulate a viscoelastic fluid. He found that the equations indeed do describe a viscoelastic fluid provided the following inequalities hold true:

$$\begin{gathered}\lambda_1\mu_0+(\lambda_1-\tfrac{3}{2}\mu_0)\nu_1 > \lambda_2\mu_0+(\lambda_1-\tfrac{3}{2}\mu_0) \geqq \tfrac{1}{9}[\lambda_1\mu_0+(\lambda_1-\tfrac{3}{2}\mu_0)\nu_1]\,,\\ (\lambda_1-\tfrac{3}{2}\mu_0)(\lambda_1\nu_2-\lambda_2\nu_1) \geqq 0\,.\end{gathered} \tag{6}$$

It is convenient at this stage to reduce the equations of state to non-dimensional form. We choose the bubble radius as our standard length and the speed of the fluid at infinity as our standard speed. We therefore make our equations dimensionless by dividing all stresses by $\eta_0 U_0/a$, all velocities by U_0 and all distances by a. The equation of state then becomes

$$P_{ik}+\varepsilon\left[\frac{\mathbb{D}}{\mathbb{D}t}P_{ik}+\beta P_{jj}e_{ik}+\alpha P_{jl}P_{jl}g_{ik}\right] = 2\left[e_{ik}+\varepsilon\left\{\eta\frac{\mathbb{D}}{\mathbb{D}t}e_{ik}+\omega e_{jl}e_{jl}g_{ik}\right\}\right], \tag{7}$$

where

$$\eta = \frac{\lambda_2}{\lambda_1},\ \beta = \frac{\mu_0}{\lambda_1},\ \alpha = \frac{\nu_1}{\lambda_1},\ \omega = \frac{\nu_2}{\lambda_1},\ \varepsilon = \frac{U_0\lambda_1}{a}\,.$$

Since we are concerned with a fluid which is only slightly non-Newtonian we therefore expect our equations to give the stress and strain relationship for a Newtonian in a suitable limit. Consider a perturbation in the parameter ε. As ε tends to zero equation (7) reduces to

$$P_{ik} = 2e_{ik}\,,$$

which is the stress-strain relation for a purely viscous fluid. It is therefore plausible to attempt a regular perturbation in ε. We shall later impose the creeping flow condition, viz: $\mathrm{Re} \ll 1$ (Reynolds number). Consequently, the solution obtained here will be valid only if ε is less than unity and at the same time much greater than Re. In other words the physical problem must satisfy the double condition

$$\frac{U_0 a\rho}{\eta_0} \ll \frac{U_0\lambda_1}{a} < 1\,, \tag{8}$$

where ρ is the density of the viscoelastic fluid.

It is physically possible to realise these conditions. For any given fluid the first inequality imposes a restriction on the size of the bubble. The radius of the bubble must be much less than $(\lambda_1\eta_0/\rho)^{\frac{1}{2}}$. And for a given fluid and bubble radius the second condition imposes a limit on the speed of the bubble. The mathematical problem then is to find the solution of the Navier-Stokes equations

$$\frac{\partial}{\partial x_k} S_{ik} = \mathrm{Re}\, u_k \frac{\partial u_i}{\partial x_k} \tag{9}$$

where S_{ik} is given by (2) and (7). And finally if the fluid is taken to be incompressible:

$$e_{ii} = 0\,. \tag{10}$$

3. Internal motion

The bubble fluid is assumed to have viscosity $\hat{\eta}_0$, density $\hat{\rho}$ and to be incompressible. The fluid motion is creeping. The internal fluid motion is therefore governed by the equation

$$\nabla \hat{P} = R\nabla^2 \hat{u} \tag{11}$$

$$\nabla \cdot \hat{u} = 0 \tag{12}$$

where $R = \hat{\eta}_0/\eta_0$, $\hat{u}$ is the dimensionless velocity and $\hat{P}$ the non-dimensional isotropic pressure.

4. Boundary conditions

We shall assume that the two fluid phases are immiscible and free of surfactants and have constant surface tension. We assume further that in the perturbed state the equation of the drop may be written as $r = 1 + \gamma f(\theta)$, i.e. the bubble is supposed to be only slightly different from spherical. Then the following conditions must be satisfied on the surface of the bubble:

(1) Normal velocity components vanish,
(2) Tangential velocities must be continuous,
(3) Tangential shear stresses are continuous, (13)
(4) Normal stress components must be discontinuous by an amount

$$\frac{\sigma}{U_0 \eta_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \text{i.e.} \quad N = \hat{N} + \frac{\sigma}{\eta_0 U_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where N is the dimensionless normal stress, R_1, R_2 are principal radii of curvature of the drop surface and σ is the coefficient of surface tension,

(5) The velocity at infinity is U_0 in the $\theta = 0$ direction.

5. Equations of motion

The equations of continuity may be satisfied by introducing the stream function ψ for the incipient fluid such that

$$u_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{\mathrm{r}\sin\theta}\frac{\partial\psi}{\partial r}. \tag{14}$$

A similar stream function $\hat{\psi}$ is defined for the fluid internal to the bubble.

The equations of motion may be written in the form

$$\frac{\partial P}{\partial r} = \frac{\partial P_{rr}}{\partial r} + \frac{1}{r}\frac{\partial P_{r\theta}}{\partial\theta} + \frac{1}{r}\{2P_{rr} - P_{\theta\theta} - P_{\phi\phi} + P_{r\theta}\cot\theta\}\,, \tag{15}$$

$$\frac{\partial P}{\partial\theta} = r\frac{\partial P_{r\theta}}{\partial r} + \frac{\partial P_{\theta\theta}}{\partial\theta} + (P_{\theta\theta} - P_{\phi\phi})\cot\theta + 3P_{r\theta}\,, \tag{16}$$

since, on account of rotational symmetry, $\partial/\partial\phi = 0$.

The equation of state may, similarly, be written as

$$\begin{aligned} &P_{rr} + \varepsilon\left[u_r\frac{\partial P_{rr}}{\partial r} + \frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}P_{rr} - 2P_{r\theta}\right) + 2P_{r\theta}(w_{r\theta} - e_{r\theta}) - 2e_{rr}P_{rr}\right] + \\ &\quad + \beta\varepsilon e_{rr}(P_{rr} + P_{\theta\theta} + P_{\phi\phi}) + \alpha\varepsilon(e_{rr}P_{rr} + e_{\theta\theta}P_{\theta\theta} + e_{\phi\phi}P_{\phi\phi} + 2e_{r\theta}P_{r\theta}) = \\ &= 2\left[e_{rr} + \eta\varepsilon\left[u_r\frac{\partial}{\partial r}e_{rr} + \frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}e_{rr} - 2e_{r\theta}\right) + 2e_{r\theta}(w_{r\theta} - e_{r\theta}) - 2e_{rr}^2\right] + \right. \\ &\quad \left. + \omega\varepsilon(e_{rr}^2 + e_{\theta\theta}^2 + e_{\phi\phi}^2 + 2e_{r\theta}^2)\right], \end{aligned} \tag{17}$$

$$P_{\theta\theta}+\varepsilon\left[u_r\frac{\partial}{\partial r}P_{\theta\theta}+\frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}P_{\theta\theta}+2P_{r\theta}\right)-2P_{r\theta}(w_{r\theta}+e_{r\theta})-2e_{\theta\theta}P_{\theta\theta}\right]+$$
$$+\beta\varepsilon e_{\theta\theta}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})+\alpha\varepsilon(e_{rr}P_{rr}+e_{\theta\theta}P_{\theta\theta}+e_{\phi\phi}P_{\phi\phi}+2e_{r\theta}P_{r\theta})=$$
$$=2\left[e_{\theta\theta}+\eta\varepsilon\left[u_r\frac{\partial}{\partial r}e_{\theta\theta}+\frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}e_{\theta\theta}+2e_{r\theta}\right)-\right.\right.$$
$$\left.\left.-2e_{r\theta}(w_{r\theta}+e_{r\theta})-2e_{\theta\theta}^2\right]+\omega\varepsilon(e_{rr}^2+e_{\theta\theta}^2+e_{\phi\phi}^2+2e_{r\theta}^2)\right], \tag{18}$$

$$P_{\phi\phi}+\varepsilon\left[u_r\frac{\partial}{\partial r}P_{\phi\phi}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}P_{\phi\phi}-2e_{\phi\phi}P_{\phi\phi}\right]+\beta\varepsilon e_{\phi\phi}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})+$$
$$+\alpha\varepsilon(e_{rr}P_{rr}+e_{\theta\theta}P_{\theta\theta}+e_{\phi\phi}P_{\phi\phi}+2e_{r\theta}P_{r\theta})=$$
$$=2\left[e_{\phi\phi}+\eta\varepsilon\left[u_r\frac{\partial}{\partial r}e_{\phi\phi}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}e_{\phi\phi}-2e_{\phi\phi}^2\right]+\right.$$
$$\left.+\omega\varepsilon(e_{rr}^2+e_{\theta\theta}^2+e_{\phi\phi}^2+2e_{r\theta}^2)\right], \tag{19}$$

$$P_{r\theta}+\varepsilon\left[u_r\frac{\partial}{\partial r}P_{r\theta}+\frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}P_{r\theta}+P_{rr}-P_{\theta\theta}\right)+P_{\theta\theta}(w_{r\theta}-e_{r\theta})-\right.$$
$$\left.-P_{rr}(w_{r\theta}+e_{r\theta})-P_{r\theta}(e_{rr}+e_{\theta\theta})\right]+\beta\varepsilon e_{r\theta}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})=$$
$$=2\left[e_{r\theta}+\eta\varepsilon\left[u_r\frac{\partial}{\partial r}e_{r\theta}+\frac{u_\theta}{r}\left(\frac{\partial}{\partial\theta}e_{r\theta}+e_{rr}-e_{\theta\theta}\right)\right]+\right.$$
$$\left.+e_{\theta\theta}(w_{r\theta}-e_{r\theta})-e_{rr}(w_{r\theta}+e_{r\theta})-e_{r\theta}(e_{rr}+e_{\theta\theta})\right]. \tag{20}$$

In order to permit an expansion in the parameter ε it is assumed that the quantities

$$u_i,\ e_{ik},\ P_{ik},\ \psi,\ \hat{\psi},\ w_{ik},\ P,\ \hat{P} \tag{21}$$

may be written in the form

$$A=\sum_{i=0}^{\infty}\varepsilon^i A^{(i)}\,. \tag{22}$$

Writing (21) in the form of (22) and substituting into equations (17)–(20) it is then possible to write

$$P_{ik}^{(m)}=2e_{ik}^{(m)}+f_{ik}^{(m)}\,,\qquad m\geqq 0\,, \tag{23}$$

by equating to zero the coefficient of ε^m in the resulting equations. Generally, the stream function $\psi^{(i+1)}$ satisfies the equation

$$E^4\psi^{(i+1)}=\sin\theta\left(\frac{\partial f_1^{(i)}}{\partial\theta}-\frac{\partial f_2^{(i)}}{\partial r}\right), \tag{24}$$

where

$$E^2\equiv\frac{\partial^2}{\partial r^2}+\frac{(1-\cos\theta)^2}{r^2}\frac{\partial^2}{\partial(\cos\theta)^2}, \tag{25}$$

$$f_1^{(i-1)}=\frac{\partial}{\partial r}f_{rr}^{(i-1)}+\frac{1}{r}\frac{\partial}{\partial\theta}f_{r\theta}^{(i-1)}+\frac{1}{r}(2f_{rr}^{(i-1)}-f_{\theta\theta}^{(i-1)}+f_{\phi\phi}^{(i-1)}+f_{r\theta}^{(i-1)}\cot\theta)\,, \tag{26}$$

$$f_2^{(i-1)}=r\frac{\partial}{\partial r}f_{r\theta}^{(i-1)}+\frac{\partial}{\partial\theta}f_{\theta\theta}^{(i-1)}+(f_{\theta\theta}^{(i-1)}-f_{\phi\phi}^{(i)})\cot\theta+3f_{r\theta}^{(i-1)}\,. \tag{27}$$

Specifically,

$$E^4\psi^{(0)} = E^4\hat{\psi}^{(0)} = 0\,. \tag{28}$$

It is evident that the boundary conditions (13) will be satisfied if each pair $\psi^{(i)}$, $\hat{\psi}^{(i)}$ satisfies the boundary conditions (13), and $\psi^{(i)}$ tends to zero as r tends to infinity for $i \geqq 1$. The solutions of (28) satisfying the pertinent boundary conditions are quite well-known, and are simply quoted here:

$$\psi^{(0)} = \left\{\frac{2+3R}{4(1+R)}r - \frac{R}{4(1+R)}\frac{1}{r} - \frac{r^2}{2}\right\}\sin^2\theta\,, \tag{29}$$

$$\hat{\psi}^{(0)} = \frac{(r^2-r^4)}{4(1+R)}\sin^2\theta\,. \tag{30}$$

6. First order perturbation

The solution just obtained may now be used to calculate $f_{\theta\theta}^{(0)}$, $f_{r\theta}^{(0)}$, $f_{\phi\phi}^{(0)}$, $f_{rr}^{(0)}$, $f_1^{(0)}$ and $f_2^{(0)}$. After some tedious computations we obtain

$$\begin{aligned} f_{\theta\theta}^{(0)} = 2\Big[(\eta-1)\Big\{&\Big(\frac{2a_2}{r^3} - \frac{6a_5}{r^4} + \frac{12a_3}{r^5} - \frac{40a_4}{r^6} - \frac{42a_6}{r^8}\Big)\cos^2\theta + \\ &+\Big(-\frac{a_2}{r^3} + \frac{a_5}{r^4} - \frac{9a_3}{r^5} + \frac{14a_4}{r^6} - \frac{27a_6}{r^8}\Big)\sin^2\theta\Big\} + \\ &+(\omega-\alpha)\Big\{\Big(\frac{6a_5}{r^4} + \frac{36a_4}{r^6} + \frac{54a_6}{r^8}\Big)\cos^2\theta + \frac{18a_6}{r^8}\sin^2\theta\Big\}\Big], \end{aligned} \tag{31}$$

$$f_{r\theta}^{(0)} = 2(\eta-1)\left\{-\frac{3a_2}{r^3} + \frac{6a_5}{r^4} - \frac{24a_3}{r^5} + \frac{36a_4}{r^6} - \frac{6a_6}{r^8}\right\}\sin\theta\cos\theta\,, \tag{32}$$

$$\begin{aligned} f_{\phi\phi}^{(0)} = 2\Big[(\eta-1)\Big\{&\Big(\frac{2a_2}{r^3} - \frac{6a_5}{r^4} + \frac{12a_3}{r^5} - \frac{40a_4}{r^6} - \frac{42a_6}{r^8}\Big)\cos^2\theta + \\ &+\Big(-\frac{a_2}{r^3} + \frac{a_5}{r^4} - \frac{3a_3}{r^5} + \frac{2a_4}{r^6} - \frac{3a_6}{r^8}\Big)\sin^2\theta\Big\} + \\ &+(\omega-\alpha)\Big\{\Big(\frac{6a_5}{r^4} + \frac{36a_4}{r^6} + \frac{54a_6}{r^8}\Big)\cos^2\theta + \frac{18a_6}{r^8}\sin^2\theta\Big\}\Big], \end{aligned} \tag{33}$$

$$\begin{aligned} f_{rr}^{(0)} = 2\Big[(\eta-1)\Big\{&\Big(\frac{-4a_2}{r^3} - \frac{24a_3}{r^5} + \frac{8a_4}{r^6} - \frac{a_6}{r^8}\Big)\cos^2\theta + \\ &+\Big(\frac{2a_2}{r^3} - \frac{4a_5}{r^4} + \frac{12a_3}{r^5} - \frac{16a_4}{r^6} - \frac{6a_6}{r^8}\Big)\sin^2\theta\Big\} + \\ &+(\omega-\alpha)\Big\{\Big(\frac{6a_5}{r^4} + \frac{36a_4}{r^6} + \frac{54a_6}{r^8}\Big)\cos^2\theta + \frac{18a_6}{r^8}\sin^2\theta\Big\}\Big], \end{aligned} \tag{34}$$

$$\begin{aligned} f_1^{(0)} = 2(\eta-1)\Big\{&\cos^2\theta\Big(\frac{-6a^2}{r^4} + \frac{24a_5}{r^5} + \frac{120a_4}{r^7} + \frac{216a_6}{r^9}\Big) + \\ &+\sin^2\theta\Big(\frac{3a_2}{r^4} - \frac{4a_5}{r^5} + \frac{12a_4}{r^7} + \frac{72a_6}{r^9}\Big)\Big\} + \\ &+(\omega-\alpha)\Big[\Big(\frac{-24a_5}{r^5} - \frac{216a_4}{r^7} - \frac{432a_6}{r^9}\Big)\cos^2\theta - \frac{144a_6}{r^9}\sin^2\theta\Big], \end{aligned} \tag{35}$$

$$f_2^{(0)} = 2(\eta-1)\left\{\frac{-6a_2}{r^3} + \frac{8a_5}{r^4} + \frac{12a_4}{r^6} + \frac{36a_6}{r^8}\right\}\sin\theta\cos\theta$$

$$-2(\omega-\alpha)\left[\frac{12a_5}{r^4} + \frac{72a_4}{r^6} + \frac{72a_6}{r^8}\right]\sin\theta\cos\theta\,, \tag{36}$$

where

$$a_2 = \frac{2+3R}{4(1+R)},\quad a_3 = \frac{-R}{4(1+R)},\quad a_4 = \frac{-R(2+3R)}{16(1+R)^2},$$

$$a_5 = \frac{1}{16}\left(\frac{2+3R}{1+R}\right)^2,\quad a_6 = \frac{1}{16}\left(\frac{R}{1+R}\right)^2.$$

From (24) it may be shown that $\psi^{(1)}$ satisfies the equation

$$E^4\psi^{(1)} = 2(\eta-1)\left\{\frac{9R(2+3R)}{(1+R)^2}\frac{1}{r^7} - \frac{3}{2}\left(\frac{3R+2}{1+R}\right)^2\frac{1}{r^5}\right\}\sin^2\theta\cos\theta\,, \tag{37}$$

and by the use of (11), (14) and (21) we may also show that

$$E^4\hat{\psi}^{(1)} = 0\,. \tag{38}$$

Solutions of (37) and (38) satisfying the relevant boundary conditions may be shown to be

$$\psi^{(1)} = \frac{(\eta-1)(2+3R)}{8(1+R)^2}\left\{\frac{2+3R}{r} + \frac{R}{r^3} - \frac{(5R^2+10R+6)}{5(1+R)} - \right.$$

$$\left. - \frac{(15R^2+20R+4)}{5(1+R)}\frac{1}{r^2}\right\}\sin^2\theta\cos\theta\,, \tag{39}$$

$$\hat{\psi}^{(1)} = \frac{(3+2R)}{40(1+R)^3}(\eta-1)(r^3-r^5)\sin^2\theta\cos\theta\,. \tag{40}$$

We have now shown that to the first order in ε the velocity distributions within the bubble and in the surrounding fluid are given by

$$\hat{\psi} = \frac{(r^2-r^4)\sin^2\theta}{4(1+R)} + \frac{\varepsilon(3+2R)(\eta-1)(r^3-r^5)}{40(1+R)^3}\sin^2\theta\cos\theta\,, \tag{41}$$

$$\psi = \left(\frac{2+3R}{4(1+R)}r - \frac{R}{4(1+R)}\frac{1}{r} - \frac{r^2}{2}\right)\sin^2\theta +$$

$$+ \frac{\varepsilon(2+3R)}{8(1+R)^2}(\eta-1)\left[\frac{2+3R}{r} + \frac{R}{r^3} - \frac{(5R^2+10R+6)}{5(1+R)} - \right.$$

$$\left. \frac{(15R^2+20R+4)}{5(1+R)}\frac{1}{r^2}\right]\sin^2\theta\cos\theta\,. \tag{42}$$

7. Drag and deformation

On account of symmetry the second term in (42) will contribute nothing to the drag on the bubble and consequently the drag on the bubble is the well-known Hadamard-Rybcyznski drag. However, higher order terms which are not considered in this work may contribute to the drag on the body.

We may now employ the conditions (13) to calculate the deformation of the bubble. For small deformation, $\gamma \ll 1$, $R_1^{-1}+R_2^{-1}$ may be replaced by (Landau & Lifshitz [8])

$$2-2f - \frac{d}{d\mu}\left[(1-\mu^2)\frac{df}{d\mu}\right].$$

Consequently the bubble deformation f is determined by the equation

$$2-2f-\frac{d}{d\mu}\left[(1-\mu^2)\frac{df}{d\mu}\right]=\frac{-\varepsilon(\eta-1)\eta_0 u_0\{180R^3+240R^2+816R+672\}P_2(\mu)}{240\sigma(1+R)^3} \tag{43}$$

where $\mu=\cos\theta$, $P_n(\mu)$ are Legendre's functions. This equation is to be solved subject to the conditions that the volume of the bubble remains constant and that angular momentum of the drop is conserved, i.e.,

$$\int_{-1}^{1} f d\mu=\int_{-1}^{1} \mu f d\mu=0\,. \tag{44}$$

It may be shown that the solution of (43) satisfying (44) is given by

$$f=\frac{u_0^2\eta_0}{a\sigma}(\lambda_1-\lambda_2)\frac{(180R^3+240R^2+816R+672)P_2(\mu)}{960(1+R)^3}. \tag{45}$$

Equation (45) shows, clearly, that the bubble deformation is towards a spheroidal shape which may be oblate or prolate according as the coefficient of $P_2(\mu)$ is negative or positive. To the order of approximation considered herein, the nature of the shape of the bubble depends solely on the difference between retardation and relaxation time. For the fluid under consideration this difference is always positive and so the bubble always takes a prolate spheroidal shape. It is clear that when $\lambda_1=\lambda_2$ (the fluid is purely viscous) the bubble remains perfectly spherical.

We have shown that while a bubble moving at zero Reynolds number in a purely viscous fluid remains perfectly spherical its shape will depart from spherical when it moves in a viscoelastic fluid. One must therefore expect the dynamics of bubbles in a viscoelastic fluid to be significantly different from their motion in a purely viscous fluid.

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